Steady-state analysis of a bistable system with additive and multiplicative noises

Ya Jia
China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China
*Department of Physics, Huazhong Normal University, Wuhan 430070, People's Republic of China**

Jia-rong Li
Institute of Particle Physics, Huazhong Normal University, Wuhan 430070, People's Republic of China
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An approximate Fokker-Planck equation for a general one-dimensional system driven by correlated noises is derived; the correlation times of the correlations between the noises are nonzero. The steady-state properties of the bistable kinetic model are analyzed. We find the following. (1) In the α-D parameter plane (α and D are the additive noise and multiplicative noise intensities, respectively), the area of the bimodal region of the stationary probability distribution (SPD) is contracted as λ is increased (λ is the strength of the correlations between noises), but the area of the bimodal region of the SPD is enlarged as τ is increased (τ is the correlation time of the correlations between noises). (2) λ and τ play opposing roles in the transition of the SPD of the system. (3) For the case of perfectly correlated noises ($\lambda=1$), there is not the phenomenon of the critical ratio ($\alpha/D=1$) which was shown by Wu, Cao, and Ke [Phys. Rev. E **50**, 2496 (1994)]. (4) The change of the mean of the state variable is very remarkable in the small τ and large λ regimes. (5) The normalized variance of the state variable increases with increasing τ, but decreases with increasing λ. [S1063-651X(96)10205-9]

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I. INTRODUCTION

Recently, the study of dynamical systems with correlation noise terms has attracted attention in the field of stochastic systems [1–9]. On the level of the Langevin-type description of dynamical systems, the presence of correlations between the noises changes the dynamics of the system. A typical case with correlation noise terms is described by the following stochastic differential equation:

$$\dot{x}(t)=f(x)+g_1(x)\xi(t)+g_2(x)\eta(t), \tag{1}$$

where $\xi(t)$ and $\eta(t)$ are Gaussian white noises with zero mean, and

$$\begin{aligned}\gamma_{11}(t,t')&=\langle\xi(t)\xi(t')\rangle=2D\delta(t-t'),\\ \gamma_{22}(t,t')&=\langle\eta(t)\eta(t')\rangle=2\alpha\delta(t-t').\end{aligned} \tag{2}$$

Most previous studies have been based on the assumption that correlations between $\xi(t)$ and $\eta(t)$ are proportional to Dirac delta functions of $(t-t')$ [4–9]:

$$\langle\xi(t)\eta(t')\rangle=\langle\eta(t)\xi(t')\rangle=2\lambda\sqrt{\alpha D}\,\delta(t-t') \quad (0\leqslant\lambda\leqslant 1). \tag{3}$$

That is, the correlation times of the correlations between $\xi(t)$ and $\eta(t)$ are zero. However, in certain situations the correlation times of the correlations between $\xi(t)$ and $\eta(t)$ may be nonzero.

It must be pointed out that Fox has discussed N-component stochastic processes with correlations between the noises in Ref. [1], where the correlation times of the correlations between noises are nonzero. He obtained an explicit equation for the probability distribution to first order in the correlation times by making use of the method of ordered operator cumulants.

In this paper, we will discuss the stochastic system (1) with (2), and assume that the correlation times of the correlations between $\xi(t)$ and $\eta(t)$ are nonzero. Here we assume

$$\begin{aligned}\gamma_{12}(t,t')&=\langle\xi(t)\eta(t')\rangle=\gamma_{21}(t,t')=\langle\eta(t)\xi(t')\rangle\\ &=\frac{\lambda\sqrt{\alpha D}}{\tau}\exp[-|t-t'|/\tau]\\ &\rightarrow 2\lambda\sqrt{\alpha D}\,\delta(t-t') \quad \text{as} \quad \tau\rightarrow 0,\end{aligned} \tag{4}$$

where τ is the correlation time of the correlations between $\xi(t)$ and $\eta(t)$, and λ is the strength of the correlations between $\xi(t)$ and $\eta(t)$. Attention is restricted here to the steady-state regime of the stochastic system. The paper is arranged as follows. In Sec. II the approximate Fokker-Planck equation (AFPE) for the general one-dimensional system (1) with (2) and (4) will be derived. By virtue of the AFPE, we study the single bistable kinetic process driven by correlated additive and multiplicative noise with nonzero correlation time, and obtain the steady-state distribution functions of the system in Sec. III. Finally, several conclusions are given in Sec. IV.

II. APPROXIMATE FOKKER-PLANCK EQUATION

A general equation satisfied by the probability distribution of the process (1) with (2) and (4) is given by [10,7]

$$\begin{aligned}\frac{\partial}{\partial t}P(x,t)=&-\frac{\partial}{\partial x}f(x)P(x,t)-\frac{\partial}{\partial x}g_1(x)\langle\xi(t)\delta(x(t)-x)\rangle\\ &-\frac{\partial}{\partial x}g_2(x)\langle\eta(t)\delta(x(t)-x)\rangle,\end{aligned} \tag{5}$$

*Mailing address.

where

$$P(x,t)=\langle \delta(x(t)-x)\rangle. \tag{6}$$

The average in (5) may be calculated for Gaussian noises $\eta(t)$ and $\xi(t)$ by a functional formula, the Novikov theorem [11]:

$$\langle \zeta_k \Phi[\zeta_1,\zeta_2]\rangle=\int_0^t dt'\,\gamma_{kl}\,\frac{\delta(\delta(x(t)-x))}{\delta\zeta_l},\quad k,l=1,2, \tag{7}$$

where $\Phi[\zeta_1,\zeta_2]$ is a functional of ζ_1 and ζ_2 and $\gamma_{kl}=\langle \zeta_k(t)\zeta_l(t')\rangle$ are its correlation functions. In our situation, ζ_1 and ζ_2 are the noises $\xi(t)$ and $\eta(t)$. According to (7) we have

$$\begin{aligned}\langle \xi(t)\delta(x(t)-x)\rangle&=\int_0^t dt'\,\gamma_{11}(t,t')\,\frac{\delta(\delta(x(t)-x))}{\delta\xi(t')}+\int_0^t dt'\,\gamma_{21}(t,t)\,\frac{\delta(\delta(x(t)-x))}{\delta\eta(t')}\\&=-D\frac{\partial}{\partial x}g_1(x)P(x,t)-\frac{\lambda\sqrt{\alpha D}}{\tau}\frac{\partial}{\partial x}\int_0^t dt'\ \exp[-|t-t'|/\tau]\\&\quad\times\left\langle \delta(x(t)-x)\,\frac{\delta x(t)}{\delta\eta(t')}\right\rangle.\end{aligned} \tag{8}$$

The response function $\delta x(t)/\delta\eta(t')$ in (8) can be given from Eq. (1):

$$\frac{\delta x(t)}{\delta\eta(t')}=g_2(x(t'))\exp\left[\int_{t'}^t ds[f'(x(s))+g_1'(x(s))\xi(s)+g_2'(x(s))\eta(s)]\right], \tag{9}$$

in which f', g_1', and g_2' denote the first x derivatives of f, g_1, and g_2, respectively. Substituting (9) into (8), we get

$$\begin{aligned}\langle \xi(t)\delta(x(t)-x)\rangle=&-D\frac{\partial}{\partial x}g_1(x)P(x,t)\\&-\frac{\lambda\sqrt{\alpha D}}{\tau}\frac{\partial}{\partial x}\int_0^t dt'\ \exp[-|t-t'|/\tau]\\&\times\left\langle g_2(x(t'))\delta(x(t)-x)\right.\\&\times\exp\left[\int_{t'}^t ds[f'(x(s))+g_1'(x(s))\xi(s)\right.\\&\left.\left.+g_2'(x(s))\eta(s)]\right]\right\rangle.\end{aligned} \tag{10}$$

Following Fox's approach [12], we observe that

$$\begin{aligned}\frac{d}{dt}g_2(x(t))&=g_2'(x(t))\frac{d}{dt}x(t)\\&=g_2'(x(t))[f(x(t))+g_1(x(t))\xi(t)+g_2(x(t))\eta(t)]\\&=\frac{g_2'(x(t))}{g_2(x(t))}[f(x(t))+g_1(x(t))\xi(t)+g_2(x(t))\eta(t)]g_2(x(t)).\end{aligned} \tag{11}$$

This yields the formal expression

$$g_2(x(t'))=g_2(x(t))\exp\left[\int_t^{t'} ds\,\frac{g_2'(x(s))}{g_2(x(s))}\times[f(x(s))+g_1(x(s))\xi(s)+g_2(x(s))\eta(s)]\right]. \tag{12}$$

When this expression is inserted into (10), we obtain

$$\begin{aligned}\langle \xi(t)\delta(x(t)-x)\rangle=&-D\frac{\partial}{\partial x}g_1(x)P(x,t)-\frac{\lambda\sqrt{\alpha D}}{\tau}\frac{\partial}{\partial x}g_2(x(t))\int_0^t dt'\ \exp[-|t-t'|/\tau]\\&\times\left\langle \delta(x(t)-x)\exp\left[\int_{t'}^t ds\left[f'(x(s))+g_1'(x(s))\xi(s)-\frac{g_2'(x(s))}{g_2(x(s))}f(x(s))-\frac{g_2'(x(s))}{g_2(x(s))}g_1(x(s))\xi(s)\right]\right]\right\rangle\\\approx&-D\frac{\partial}{\partial x}g_1(x)P(x,t)-\frac{\lambda\sqrt{\alpha D}}{\tau}\frac{\partial}{\partial x}g_2(x(t))\int_0^t dt'\ \exp[-|t-t'|/\tau]\\&\times\left\langle \delta(x(t)-x)\exp\left[\int_{t'}^t ds\left(f'(x(s))-\frac{g_2'(x(s))}{g_2(x(s))}f(x(s))\right)\right]\right\rangle,\end{aligned} \tag{13}$$

where the $\xi(s)$ terms have been neglected, which can be shown to be self-consistently valid [13]. It has also been assumed that t is sufficiently large compared to τ.

In this paper we consider the stochastic process in the steady-state regime; the ansatz of Hanggi *et al.* [14] can be used here [15,16]. We argue that at steady state the last exponential in (13) is approximated by [15]

$$\exp\left[\int_{t'}^{t} ds\left(f'(x(s))-\frac{g_2'(x(s))}{g_2(x(s))}f(x(s))\right)\right]\approx\exp\left[(t-t')\left(f'(x_s)-\frac{g_2'(x_s)}{g_2(x_s)}f(x_s)\right)\right], \tag{14}$$

where x_s denotes the steady-state value of $\langle x(t)\rangle$. Inserting this approximation into (13) and performing the remaining t' integral yields

$$\langle\xi(t)\delta(x(t)-x)\rangle=-D\frac{\partial}{\partial x}g_1(x)P(x,t)-\frac{\lambda\sqrt{\alpha D}}{1-\tau[f'(x_s)-[g_2'(x_s)/g_2(x_s)]f(x_s)]}\times\frac{\partial}{\partial x}g_2(x)P(x,t). \tag{15}$$

Similarly, the average $\langle\eta(t)\delta(x(t)-x)\rangle$ in (5) can be calculated:

$$\langle\eta(t)\delta(x(t)-x)\rangle=-\alpha\frac{\partial}{\partial x}g_2(x)P(x,t)-\frac{\lambda\sqrt{\alpha D}}{1-\tau[f'(x_s)-[g_1'(x_s)/g_1(x_s)]f(x_s)]}\times\frac{\partial}{\partial x}g_1(x)P(x,t). \tag{16}$$

Substituting (15) and (16) into (5), we finally obtain the AFPE corresponding to (1) with (2) and (4):

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t}=&-\frac{\partial}{\partial x}f(x)P(x,t)+D\frac{\partial}{\partial x}g_1(x)\frac{\partial}{\partial x}g_1(x)P(x,t)\\&+\frac{\lambda\sqrt{\alpha D}}{1-\tau[f'(x_s)-[g_2'(x_s)/g_2(x_s)]f(x_s)]}\times\frac{\partial}{\partial x}g_1(x)\frac{\partial}{\partial x}g_2(x)P(x,t)\\&+\alpha\frac{\partial}{\partial x}g_2(x)\frac{\partial}{\partial x}g_2(x)P(x,t)\\&+\frac{\lambda\sqrt{\alpha D}}{1-\tau[f'(x_s)-[g_1'(x_s)/g_1(x_s)]f(x_s)]}\times\frac{\partial}{\partial x}g_2(x)\frac{\partial}{\partial x}g_1(x)P(x,t).\end{aligned} \tag{17}$$

The AFPE is valid for the following conditions:

$$1-\tau\left[f'(x_s)-\frac{g_2'(x_s)}{g_2(x_s)}f(x_s)\right]>0 \tag{18}$$

and

$$1-\tau\left[f'(x_s)-\frac{g_1'(x_s)}{g_1(x_s)}f(x_s)\right]>0; \tag{19}$$

these provide the constraint on τ.

III. STEADY-STATE DISTRIBUTION FUNCTIONS OF THE BISTABLE SYSTEM

Now we consider a popular example, the single bistable kinetic system, and assume the dimensionless form

$$\dot{x}=ax-bx^3+x\xi(t)+\eta(t)\quad(a>0,\ b>0) \tag{20}$$

with (2) and (4). This is a special case of Eq. (1) with

$$f(x)=ax-bx^3,\quad g_1(x)=x,\quad g_2(x)=1, \tag{21}$$

and the steady-state value $x_s^2=a/b$. Therefore the AFPE of the system (20) is obtained by substituting (21) into (17):

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t}=&-\frac{\partial}{\partial x}(ax-bx^3)P(x,t)+D\frac{\partial}{\partial x}x\frac{\partial}{\partial x}xP(x,t)\\&+\frac{\lambda\sqrt{\alpha D}}{1+2a\tau}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}P(x,t)+\alpha\frac{\partial^2}{\partial x^2}P(x,t)\\&+\frac{\lambda\sqrt{\alpha D}}{1+2a\tau}\frac{\partial^2}{\partial x^2}xP(x,t).\end{aligned} \tag{22}$$

Note that $1+2a\tau>0$ for all τ. Thus there exists no restriction on τ in this case. The stationary probability distribution (SPD) of the system can be obtained from (22) and is given by

$$P_{\rm st}(x)=N_1[Dx^2+2\lambda\sqrt{\alpha D}x+\alpha]^{\beta_1-1/2}\exp\left[-\frac{b}{2D}x^2+\frac{2\lambda\sqrt{\alpha D}b}{D^2}x+\frac{\lambda}{\sqrt{1-\lambda^2}}\theta_1\arctan\frac{\lambda+\sqrt{D/\alpha}x}{\sqrt{1-\lambda^2}}\right]\quad\text{for}\ \ \tau=0,\ 0\leqslant\lambda<1, \tag{23}$$

$$P_{\rm st}(x)=N_2[Dx^2+2\sqrt{\alpha D}x+\alpha]^{\beta_2-1/2}\exp\left[-\frac{b}{2D}x^2+\frac{2\sqrt{\alpha D}b}{D^2}x-\frac{\theta_2}{Dx+\sqrt{\alpha D}}\right]\quad\text{for}\ \ \tau=0,\ \lambda=1, \tag{24}$$

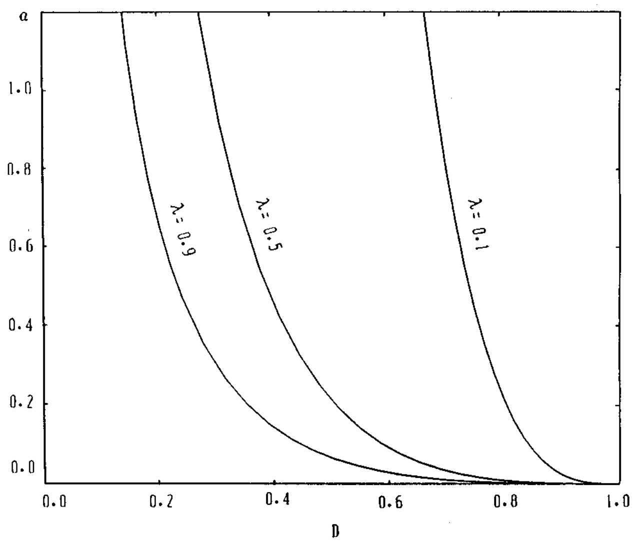

FIG. 1. The critical curves in the α-D parameter plane plotted from Eq. (30) for $\tau=0.1$. $\lambda=0.1$, 0.5, and 0.9, respectively.

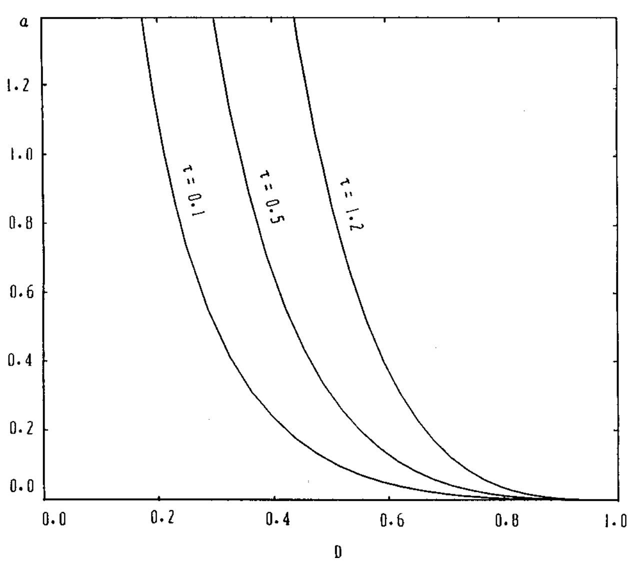

FIG. 2. The critical curves in the α-D parameter plane plotted from Eq. (30) for $\lambda=0.7$. $\tau=0.1$, 0.5, and 1.2, respectively.

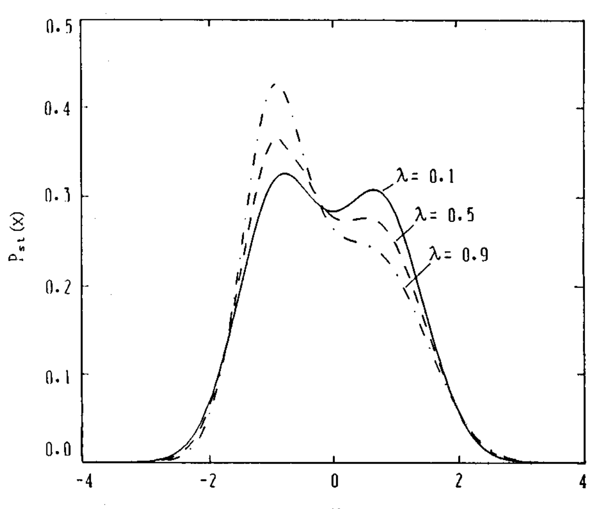

FIG. 3. The SPD of the bistable kinetic model Eq. (25) for $\alpha=D=0.5$, $\tau=0.7$. $\lambda=0.1$, 0.5, and 0.9, respectively.

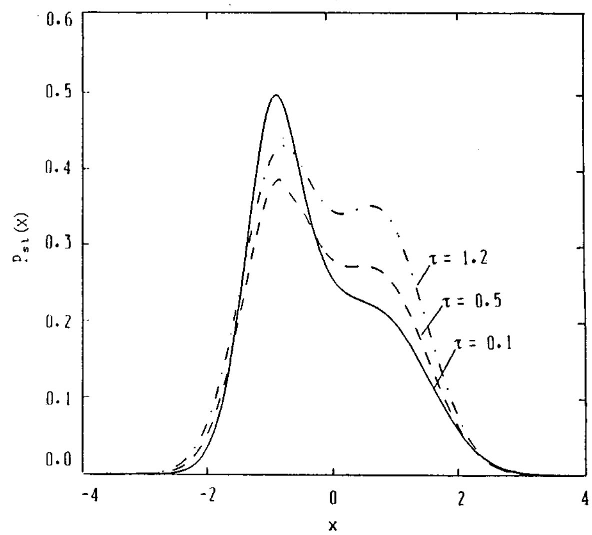

FIG. 4. The SPD of the bistable kinetic model Eq. (25) for $\alpha=D=0.5$, $\lambda=0.5$. $\tau=0.1$, 0.5, and 1.2, respectively.

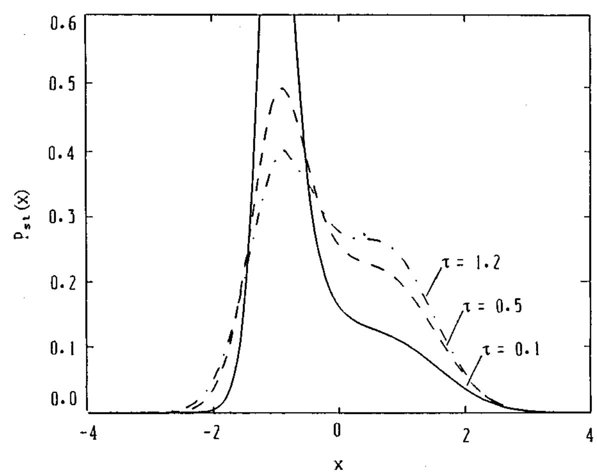

FIG. 5. The SPD of the bistable kinetic model Eq. (25) for the case of perfectly correlated noises ($\lambda=1$) with $\alpha=D=0.5$. $\tau=0.1$, 0.5, and 1.2, respectively.

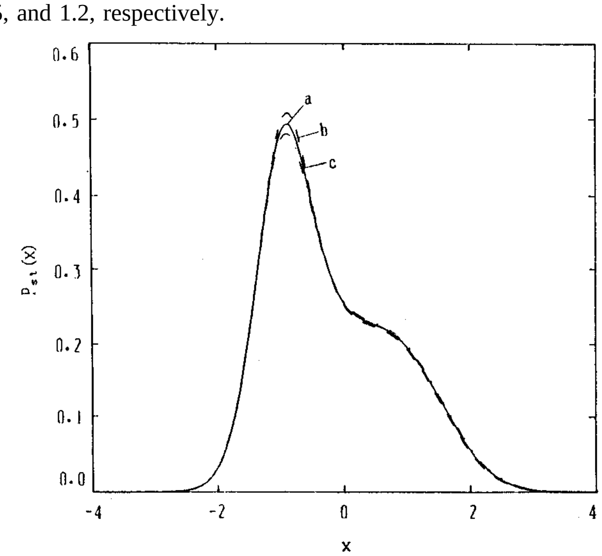

FIG. 6. The SPD of the bistable kinetic model Eq. (25) for the case of perfectly correlated noises ($\lambda=1$) with $\tau=0.5$. Curve a, the case of $\alpha/D=1$; $\alpha=0.5$, $D=0.5$. Curve b, the case of $\alpha/D<1$; $\alpha=0.45$, $D=0.5$. Curve c, the case of $\alpha/D>1$; $\alpha=0.55$, $D=0.5$.

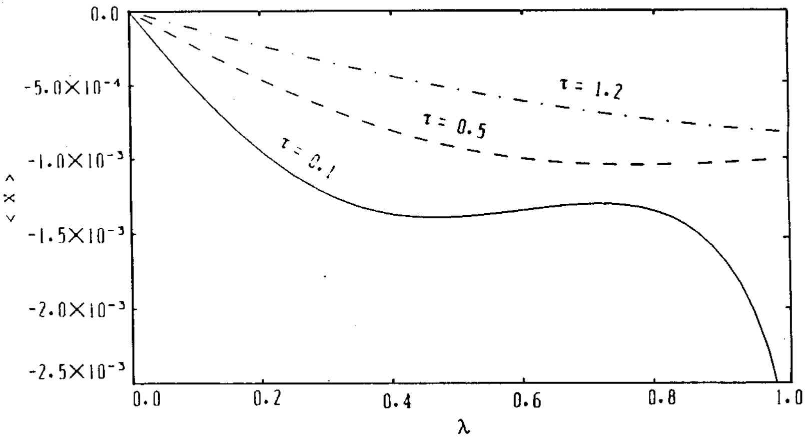

FIG. 7. The mean of the state variable $\langle x\rangle$ as a function of λ with $\alpha=D=0.5$. $\tau=0.1$, 0.5, and 1.2, respectively.

$$P_{\rm st}(x)=N_3\left[Dx^2+\frac{2\lambda\sqrt{\alpha D}x}{1+2a\tau}+\alpha\right]^{\beta_3-1/2}\exp\left[-\frac{b}{2D}x^2+\frac{2\lambda\sqrt{\alpha D}b}{D^2(1+2a\tau)}x+\frac{\lambda}{\sqrt{(1+2a\tau)^2-\lambda^2}}\theta_3\arctan\times\frac{\lambda+(1+2a\tau)\sqrt{D/\alpha}x}{\sqrt{(1+2a\tau)^2-\lambda^2}}\right]\quad\text{for }\tau\neq0,\ 0<\lambda\leqslant1,\tag{25}$$

where

$$\beta_1=\frac{(b\alpha+aD)-4b\lambda^2\alpha}{2D^2},\qquad \theta_1=\frac{4b\alpha\lambda^2-3b\alpha-aD}{D^2},\tag{26}$$

$$\beta_2=\frac{-3b\alpha+aD}{2D^2},\qquad \theta_2=\frac{\sqrt{\alpha D}(b\alpha-aD)}{D^2},\tag{27}$$

$$\beta_3=\frac{(b\alpha+aD)(1+2a\tau)^2-4b\lambda^2\alpha}{2D^2(1+2a\tau)^2},\qquad \theta_3=\frac{2b\alpha[2\lambda^2-(1+2a\tau)^2]}{D^2(1+2a\tau)^2}-\frac{a}{D}-\frac{b\alpha}{D^2},\tag{28}$$

and N_1, N_2, and N_3 are the normalization constants for Eqs. (23), (24), and (25), respectively. It must be pointed out that the correlation time τ must be zero when the strength of the correlations between noises λ is zero. Therefore the parameter λ is nonzero in Eq. (25). The extrema of the SPD (25) are determined by the following equation of third order:

$$bx^3-(a-D)x+\frac{\lambda\sqrt{\alpha D}}{1+2a\tau}=0.\tag{29}$$

The critical curve separating the unimodal and bimodal regions is determined by

$$\frac{\lambda^2\alpha D}{4(1+2a\tau)^2}-\frac{(a-D)^3}{27b}=0.\tag{30}$$

The moments of the state variable x are given by

$$\langle x^n\rangle=\int_{-\infty}^{+\infty}x^nP_{\rm st}(x)dx.\tag{31}$$

The mean and normalized variance of the state variable are given by the numerical integrations of Eq. (31). The mean of the state variable is

$$\langle x\rangle=\int_{-\infty}^{+\infty}xP_{\rm st}(x)dx\tag{32}$$

and the normalized variance of the state variable is

$$\lambda_2(0)=\frac{\langle x-\langle x\rangle\rangle^2}{\langle x\rangle^2}=\frac{\langle x^2\rangle}{\langle x\rangle^2}-1.\tag{33}$$

IV. STEADY-STATE ANALYSIS: CONCLUSIONS

When the correlation time τ is zero, the SPD's [(23) and (24)] have been discussed in Ref. [7] (in which $a=1$, $b=1$), and will not be recounted here. Our aim in this paper is to discuss the steady-state properties of the bistable system when the correlation time is nonzero. By virtue of the results Eqs. (30), (25), (32), and (33) obtained above, we have plotted the critical curves in the α-D plane in Figs. 1 and 2, the curves of the SPD in Figs. 3–6, and the curves of the mean and variance of the state variable in Figs. 7–10. Here, we take $a=1$ and $b=1$ for simplicity. The conclusions that can be drawn from these figures are as follows.

(i) The presence of correlation between the noises causes the critical curve separating the unimodal and bimodal regions in the α-D parameter plane to be affected not only by λ, the strength of the correlation between noises, but also by τ, the correlation time of the correlation between noises, as can be seen from Figs. 1 and 2.

(ii) When τ is fixed, the area of the bimodal region in Fig. 1 contracts as λ increases. However, when λ is fixed, the area

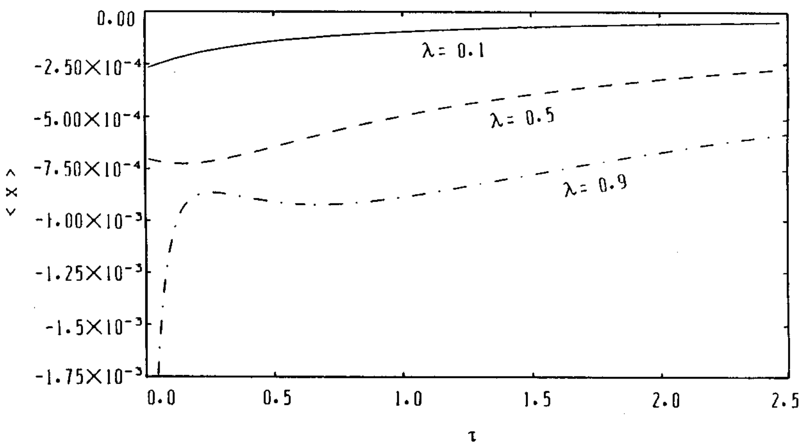

FIG. 8. The mean of the state variable $\langle x\rangle$ as a function of τ with $\alpha=D=0.5$. $\lambda=0.1$, 0.5, and 0.9, respectively.

of the bimodal region in Fig. 2 is enlarged as τ increases. This shows that the strength of the correlation between noises and the correlation time of the correlation between noises play opposing roles in the transition of the SPD.

(iii) Because of the different roles in the transition of the SPD for τ and λ, we can see that the SPD of the bistable system experiences the transition from a bimodal to a unimodal structure as λ increases (Fig. 3), but experiences the transition from a unimodal to a bimodal structure as τ increases (Fig. 4).

(iv) For the case of perfectly correlated noises ($\lambda=1$), the SPD (24) (i.e., $\tau=0$) which has been discussed in Ref. [7] exhibits divergence at $x=-(\alpha/D)^{1/2}$, and the SPD's corresponding to $\alpha/D>1$ and $\alpha/D<1$ exhibit a very different shape of divergence; therefore the ratio $\alpha/D=1$ plays the role of a critical ratio. However, when the correlation time is nonzero, the above pictures of the SPD change. The SPD (25) does not exhibits divergence in the region $-\infty<x<+\infty$ due to

$$\frac{4\alpha D}{(1+2\tau)^2}-4\alpha D<0$$

in the factor

$$\left(Dx^2+\frac{2\sqrt{\alpha D}}{1+2\tau}\,x+\alpha\right)^{\beta_3-1/2}.$$

The SPD experiences the transition from a unimodal to a bimodal structure when τ increases as shown in Fig. 5. Moreover, from Fig. 6 we can see that there is not the phenomenon of the critical ratio, namely, when the parameters α and D take values in the neighborhood of $\alpha=D$, the SPD's corresponding to $\alpha/D>1$ and $\alpha/D<1$ do not exhibit a very different shape.

(v) When the intensity of the additive noise is equal to that of the multiplicative noise, the mean of the state variable is plotted in Fig. 7 as a function of λ and as a function of τ in Fig. 8. It is obvious that the mean of the state variable is negative. The mean increases with increasing τ, but decreases with increasing λ; it is very remarkable for small τ and large λ (e.g., $\tau=0.1$, $0.9<\lambda<1.0$ in Fig. 7 and $\lambda=0.9$, $0<\tau<0.2$ in Fig. 8). The above behaviors of the mean are determined by the SPD (25) of the system according to Eq. (32).

When the noises are uncorrelated ($\lambda=0$, $\tau=0$), the SPD exhibits a symmetric bimodal structure for $\alpha=D$ [7]. However, when the noises are correlated, the symmetry of the SPD is destroyed as shown in Figs. 3–5. The SPD (25) is

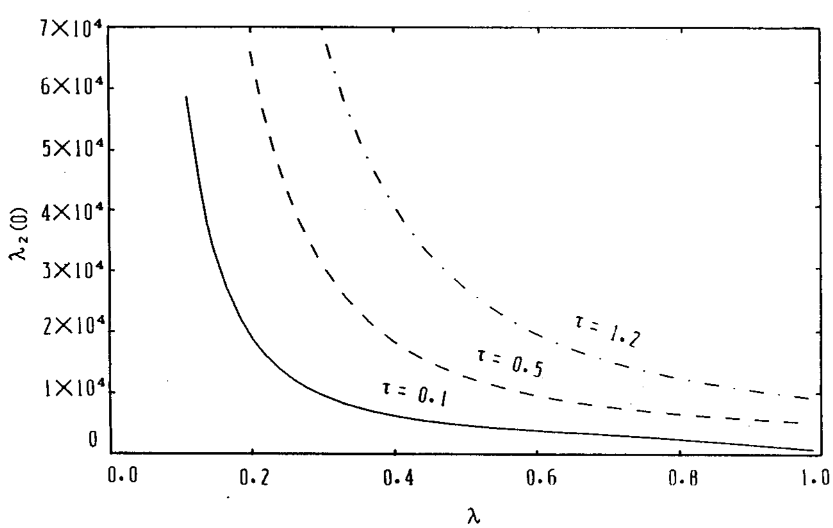

FIG. 9. The normalized variance $\lambda_2(0)$ of the state variable as a function of λ with $\alpha=D=0.5$, $\tau=0.1$, 0.5, and 1.2, respectively.

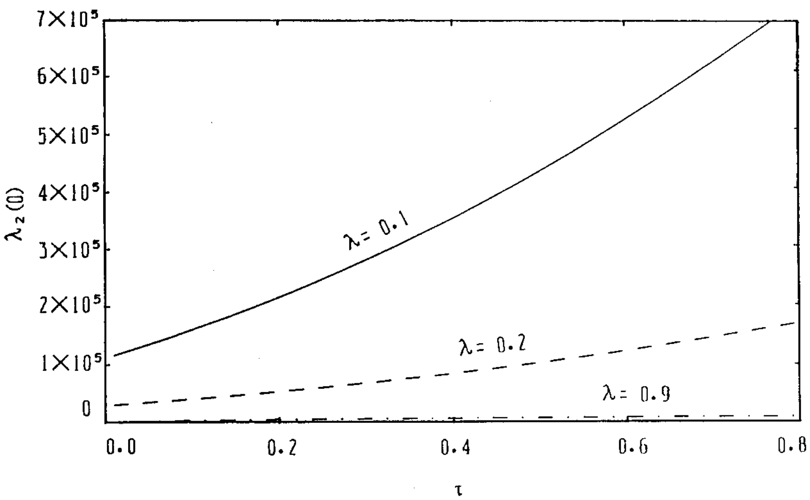

FIG. 10. The normalized variance $\lambda_2(0)$ of the state variable as a function of τ with $\alpha=D=0.5$. $\lambda=0.1$, 0.2, and 0.9, respectively.

affected not only by the intensities of noises but also by the strength of correlation between noises and the correlation time of correlation between noises. First, the presence of correlation between noises shifts the SPD to the left; the SPD appearing in $x<0$ is greater than that in $x>0$. Therefore the mean of the state variable is negative. Second, because the SPD is shifted from $x>0$ to $x<0$ by increasing λ as shown in Fig. 3, but from $x<0$ to $x>0$ by increasing τ as shown in Fig. 4, the mean of the state variable increases with increasing τ, and decreases with increasing λ. Third, because the effects of τ and λ on the change of the SPD are opposed to each other and the two effects cancel each other out, the curves of the mean (e.g., $\tau=0.5$ and 1.2 in Fig. 7 and $\lambda=0.1$ and 0.5 in Fig. 8) are smooth. However, when λ is large and τ is small, the effect of λ on the SPD plays the leading role. The SPD appears nearly completely in $x<0$ (e.g., $\lambda=1$ and $\tau=0.1$ in Fig. 5), and the extremum of the SPD shifts to the left. Therefore the mean is very remarkable for small τ and large λ (e.g., $\tau=0.1$, $0.9<\lambda<1.0$ in Fig. 7 and $\lambda=0.9$, $0<\tau<0.2$ in Fig. 8).

(vi) The normalized variance $\lambda_2(0)$ of the state variable is plotted as a function of λ in Fig. 9 and as a function of τ in Fig. 10. The normalized variance increases with increasing τ but decreases with increasing λ. This also shows that the effects of the strength of the correlation between noises and the correlation time on the normalized variance are different. In addition, because the value of the mean is very small as shown in Figs. 7 and 8, the value of the normalized variance is very large as shown in Figs. 9 and 10.

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